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EXACT DISTRIBUTIONS FOR GAPS AND STRETCHES.(U)
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EXACT DISTRIBUTIONS FOR GAPS AND STRETCHES

By

JOSEPH G. DEKEN

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EXACT DISTRIBUTIONS FOR GAPS AND STRETCHES

By

Joseph G. Deken

1. Introduction.

For random variables $X_1, X_2, \ldots, X_k$, with order statistics $Y_1, Y_2, \ldots, Y_k$, define the p-stretches $Z_1, Z_2, \ldots, Z_{k+1-p}$ as:

$$Z_j := Y_{j+p-1} - Y_j \ .$$

The variables Z_j are often called spacings $(p=2)$ or higher order spacings $(p > 2)$, and have been discussed in an extensive literature. Much of this literature deals with the classical geometric probability problem of random coverage of the circumference of a circle by random arcs. A recent article by Holst (1980) gives some new results, as well as many references. In some recent work on multiple comparisons by Welsh (1977), the variables Z_j are called gaps for $p = 2$ and stretches for $p > 2$. In the multiple comparisons context, which motivated the present work, the number of points, k, may often be small, and asymptotic results not sufficiently accurate. The key contribution is to deal directly with the lack of independence between successive p-stretches. This problem is not easily dealt with for $p > 2$, so that previous results in this case have been largely confined to asymptotics. As is shown here a recursive formulation along with symbolic computation may be used to derive exact results in many cases. Specifically, we compute here the exact distribution for the maximum p-stretch for all values of p and ten or fewer points uniformly distributed in the unit interval. The

Theorems given could be easily modified to cover densities other than the uniform.

In addition to the exact distributions, the resulting formulas for all moments, and the quantiles derived numerically to accuracy 10^{-6}, we give here some explicit moment generating functions which can be derived by the same technique. Since the distribution of any individual p-stretch is Beta, when the X's are uniform, an approximation to the distributions given is that of the maximum of $k+1-p$ independent Beta variables. Approximate quantiles based on this independence approximation are given for comparison with the exact results.

The distributions found here may be of use in a variety of multiple comparisons situations, such as those discussed by Welsch, via the appropriate probability inverse transformations. Such an approach, if sufficiently powerful, would eliminate the need for separate tables for each distribution considered. Alternatively, it may be found to be preferable to apply the techniques directly to random variables X with a specified (Gaussian or other) distribution.

In some cases, it may be of interest to consider the distances Y_1 and $1-Y_k$ as well as distances Y_j-Y_{j-1}, e.g. defining $Y_0 := 0$ and $Y_{k+1} := 1$. One would then have $k+3-p$ p-stretches $Z_0, Z_1, \ldots, Z_{k+2-p}$. The results here carry over directly by noting that $(Y_1-Y_0, Y_2-Y_1, \ldots, Y_{k+1}-Y_k) \sim \frac{1}{Y_{k+2}-Y_1}(Y_2-Y_1, \ldots, Y_{k+2}-Y_{k+1})$. Thus if $\tilde{Z}_k$ is the maximum p-stretch based on k points (without Y_0, Y_{k+1}) and Z_k^* is the maximum with Y_0 and Y_{k+1} we have for example

$$\Pr\{\tilde{Z}_k \leq t\} = \int_0^1 \Pr\{Z_{k-2}^* \leq r\,t\} \; dP\{R_k \leq r\} \, ,$$

where

$$R_k := Y_k - Y_1 \quad \text{and} \quad P\{R_k \leq r\} = kr^{k-1} - (k-1)r^k \, .$$

2. Maximum p-stretches for k points in the unit interval.

Let random variables $X_1, X_2, \ldots, X_k$ be independently and uniformly distributed on the interval $(0,s)$, with order statistics $Y_1 < Y_2 < \cdots < Y_k$. For $p \geq 2$ define the p-stretches $Z_1^{(p)}, Z_2^{(p)}, \ldots, Z_{k+1-p}^{(p)}$ as:

$$Z_j^{(p)} := Y_{j+p-1} - Y_j \ .$$

We will derive here explicit formulas for the distribution of the maximum p-stretch, $\tilde{Z}^{(p)}(x) := \max\limits_{1 \leq j \leq k+1-p} Z_j^{(p)}$. Let $E_k^{(p)}(s)$ be the event $\{\tilde{Z}^{(p)}(s) \geq 1\}$. If the probability $P\{E_k^{(p)}(s)\}$ can be given for all s, then for fixed s, all probabilities $P\{\tilde{Z}^{(p)}(s) \geq t\}$ may be obtained from the relation $P\{\tilde{Z}^{(p)}(s) \geq t\} = P\{E_k^p(s/t)\}$. Henceforth, we will restrict attention to calculating $P\{E_k^{(p)}(s)\}$.

Since the argument p will be constant throughout the following discussion, it will be omitted as a superscript, and we will write

$$P_{nk}(s) := \Pr\{E_k^{(p)}(s)\} \quad \text{when} \quad [s] = n \ .$$

The initial conditions $P_{0k}(s) = 0$, $(P_{nk}(s) = 0,\ k < p)$ are obvious. The derivation of $P_{nk}(\underset{\sim}{s})$ will utilize the following auxiliary probabilities.

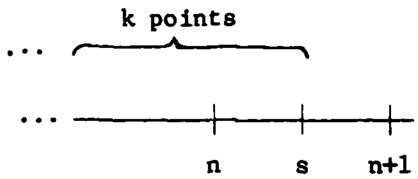
A) $P1_{nk}(\underset{\sim}{s})$, (where $\underset{\sim}{s} = (s_0, s_1, \ldots, s_{p-1})$ and $s_0 \geq [s_{p-1}] =: n$) is the probability that $E_k(s)$ occurs and there are $p-1$ points X_i in (s_0, s), at $s_1 \leq s_2 \leq \cdots \leq s_{p-1}$.

B) $P2_{nkm}(\underset{\sim}{s})$ (where $\underset{\sim}{s} = (s_0, s_1, \ldots, s_{p-1})$, $[s_{p-1}] = n$, and $n \geq s_0 \geq s_{p-1} - 1$) is the probability that $E_k(s)$ occurs and there are $p-1$ points X_i in (s_0, s), at $s_1 \leq s_2 \leq \cdots \leq s_{p-1}$, with m of these points in the interval (n, s).

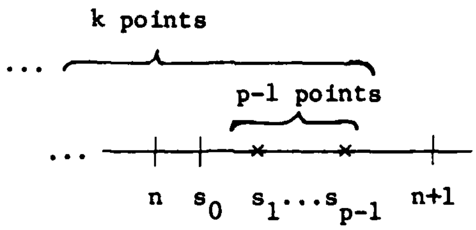
C) $AP_{nk}(t)$ is the probability that $E_k(s)$ occurs and $\max\limits_{1 \leq i \leq k} X_i = t$.

These probabilities may be illustrated graphically as:

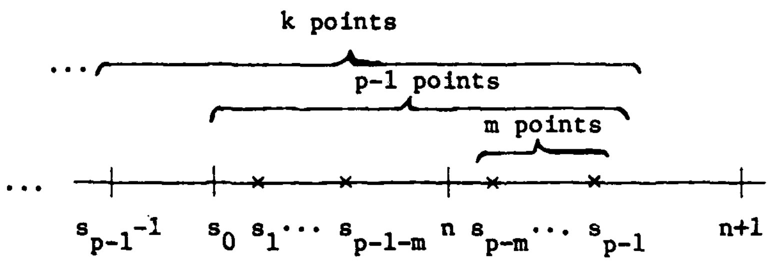
$P_{nk}(s)$:



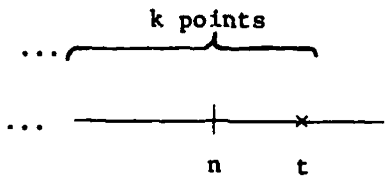
$P1_{nk}(\underset{\sim}{s})$:



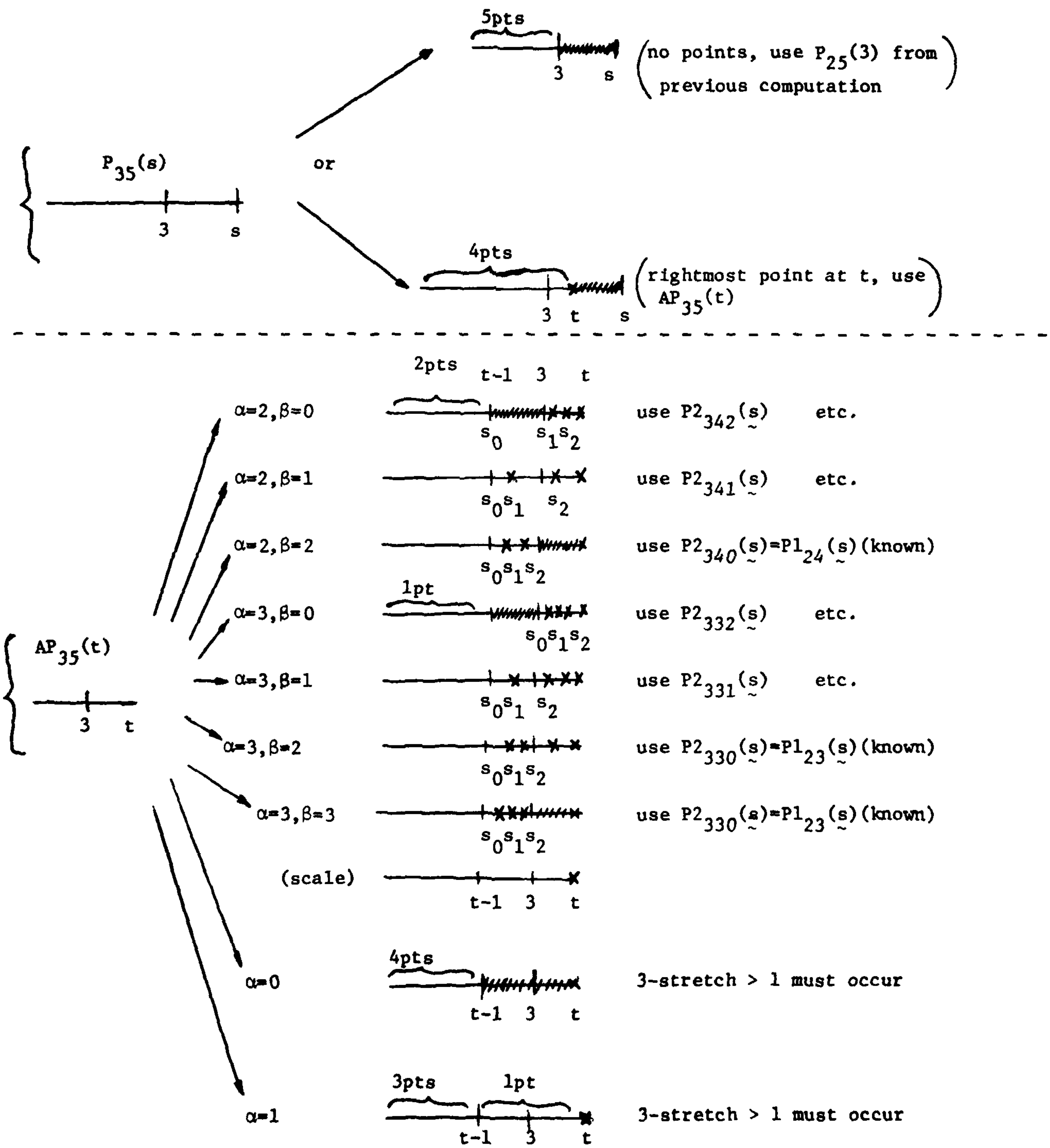
$P2_{nkm}(\underset{\sim}{s})$:



$AP_{nk}(t)$:



For example, if $n = 3$ and $k = 5$, the derivation of $P_{35}(s)$ for $p = 3$ (3 stretch) may be sketched in more detail:



With the above definitions, it is possible to compute $P_{nk}(s)$ recursively, using auxiliary probabilities AP_{nk} and $P_{n-1\,k}$. In turn, AP_{nk} depends on $(P2_{n\alpha\beta}\ \alpha \leq k,\ \beta \leq p-1)$, $P2_{n\alpha\beta}$ may be expressed in terms of $(P2_{n\alpha\gamma}\ \gamma < \beta$ and $P1_{n-1\,\delta}\ \delta < \alpha)$, and $P1_{n-1\,\delta}$ in terms of $(P2_{n-1\,\eta\,\xi}\ \eta \leq \delta,\ \xi \leq p-1)$. The practical usefulness of the recursion is enhanced by the fact that no first subscripts $< n-1$ are involved. The recursion, described in detail by the theorems below, thus has the form:

$$\begin{tikzcd} P_{nk} & P_{n-1\,k} \arrow[l] \\ AP_{nk} \arrow[u] & \\ P2_{nk\bullet} \arrow[u] & \\ P1_{n-1\,k} \arrow[u] & \\ P2_{n-1,k,\bullet} \arrow[u] & \end{tikzcd}$$

Values in solid circles are computed and saved at the n-th stage. Values in broken circles are used at the n-th stage and discarded.

Theorem 1. Using the auxiliary probabilities defined above, the following equations give P_{nk} recursively (we omit the trivial cases $n < 1$ or $k < p$ in all equations below).

$$P_{nk}(s) = E\{AP_{nk}(\max_{1 \leq i \leq k} X_i)\}$$

$$= \int_n^s AP_{nk}(t)\ dP\{\max_{1 \leq i \leq k} X_i \leq t\} + (\frac{n}{s})^k P_{n-1\,k}(n) \qquad (1)$$

$$AP_{nk}(t) = \sum_{\alpha=0}^{p-2} \Pi(k-1,t,1,\alpha)$$

$$+ \sum_{\alpha=p-1}^{k-2} \Pi(k-1,t,1,\alpha)\left(\sum_{\beta=0}^{\alpha} \Pi(\alpha,1,n-(t-1),\beta) \times\right.$$

$$\left.E\left\{P2_{n,k-1-(\alpha-(p-1)),\alpha\wedge(p-1)-\beta\wedge(p-1)}\left((t-1,\underset{\sim}{X}_{\beta,\beta\wedge(p-1)}(t-1,n)\ ,\ \underset{\sim}{X}'_{\alpha-\beta,\alpha\wedge(p-1)-\beta\wedge(p-1)}(t,n)\right)\right\}\right) \tag{2}$$

$$P2_{nkm}((t_0,t_1,\ldots,t_{p-1})) \begin{cases} = P1_{n-1\,k}(t_0,t_1,\ldots,t_{p-1}) & m = 0 \\ = \Pi(k-(p-1),t_0,t_0-(t_{p-1}-1),0) & \\ \quad + \displaystyle\sum_{\alpha=1}^{k-(p-1)-1} (k-(p-1),t_0,t_0-(t_{p-1}-1),\alpha) \times & \\ \quad E\left\{P2_{n,k-\alpha,m-(\alpha\wedge m)}\left((t_{p-1}-1,\underset{\sim}{X}_{\alpha,\alpha\wedge p-1}(t_{p-1}-1,t_0),t_1,t_2,\ldots,t_{(p-1-\alpha)_+})\right)\right\} & \\ & m > 0 \end{cases} \tag{3}$$

$$P1_{nk}(t_0,t_1,\ldots,t_{p-1}) = \Pi(k-(p-1),t_0,t_0-(t_{p-1}-1),0)$$

$$+ \sum_{\alpha=1}^{k-(p-1)-1} \Pi(k-(p-1),t_0,t_0-(t_{p-1}-1),\alpha)\Bigg(\sum_{\beta=0}^{\alpha} \Pi(\alpha,t_0-(t_{p-1}-1),n-(t_{p-1}-1),\beta)$$

$$\cdot\, E\Bigg\{P2_{n,k-\alpha,(p-1-\beta)_+}\Big[\Big(t_{p-1}-1, \underset{\sim}{X}_{\beta,\beta\wedge(p-1)}(t_{p-1}-1,n),\ X'_{\alpha-\beta,\alpha\wedge(p-1)-\beta\wedge(p-1)}(n,t_0), \underbrace{t_1,t_2,\ldots,t_{(p-1-\alpha)_+}}_{(p-1-\alpha)_+\ \text{terms}}\Big)\Big]\Bigg\}\Bigg) \tag{4}$$

<u>Proof</u>. In the above equations, $\Pi(k,s,v,j) := \binom{k}{j}v^j(s-v)^{k-j}/s^k$ is the probability that of k points uniformly distributed in an interval of length s, j lie in a subinterval of length v. Also $\underset{\sim}{X}_{jk}(a,b)$, $\underset{\sim}{X}'_{jk}(a,b)$ denote independent random vectors consisting of the first k order statistics from j points distributed uniformly in (a,b). For example, to derive (2), we note that if $\max_{1\le i\le k} X_i = t$, then if there are less than $p-1$ points in $(t-1,t)$, $E_k(s)$ occurs. Otherwise, we may condition on the positions of the leftmost $p-1$ points of the α points in $(t-1,t)$, and β points in $(t-1,n)$,

$$(\beta\wedge(p-1) \text{ of them in } (t-1,n) \text{ and } \alpha\wedge(p-1)-\beta\wedge(p-1) \text{ in } (n,t))\ ,$$

to obtain the desired probability in terms of P2. Equations (3) and (4) are derived similarly, by considering the points in $(t_{p-1}-1,t_0)$ and $(t_{p-1}-1,t_0) = (t_{p-1}-1,n)\cup(n,t_0)$ respectively. ■

3. Generating functions.

We may consider a Poisson process on the interval $(0,s)$, with rate λ and points $Y_1 < Y_2 < \cdots < Y_\nu$, so that ν is a random variable, $\Pr\{\nu=\alpha\} = e^{-\lambda s} \frac{(\lambda s)^\alpha}{\alpha!}$, and conditionally on $\nu = \alpha$, $Y_1, Y_2, \ldots, Y_\alpha$ are distributed as uniform order statistics. Then $\Pr_\lambda\{E_\nu^{(p)}(s)\} = e^{-\lambda s} \Sigma_{\alpha=p}^{\infty} \frac{(\lambda s)}{\alpha!} \Pr\{E_\alpha^{(p)}(s)\}$, where the probabilities $\Pr\{E_\alpha^{(p)}(s)\}$ are those given by Theorem 1 above. That is, $m^{(p)}(\lambda,s) := e^{\lambda s} \Pr_\lambda\{E_\nu^{(p)}(s)\}$ is the exponential generating function for the sequence $\{E_k^{(p)}(s)\}_{k=p}^{\infty}$, and may be used to give closed form expressions for $E_k^{(p)}(s)$. In particular, if $m^{(p)}(\lambda,s) = \Sigma_{j=1}^{M} c_j \lambda^{n_j} e^{\lambda a_j}$ for some non-negative integers n_j and constants (c_j, a_j), then

$$\Pr\{E_k^{(p)}(s)\} = \frac{1}{s^k} \sum_{j-1}^{M} c_j (k)_{n_j} a_j^{k-n_j} ,$$

where $(k)_j = \frac{k!}{(k-j)!}$. Our results will be of this form.

The probabilities $\Pr_\lambda\{E_\nu^{(p)}(s)\}$ may be computed recursively in exactly the fashion of Theorem 1, conditioning on some set of $j \leq p-1$ rightmost points in the interval $(0,s)$. We examine the interval $(0,s)$ from right to left and use the facts that (1) the number of points in any subinterval of length t has the Poisson distribution with parameter λt, (2) the numbers of points in any subinterval are independent random variables, and (3) conditionally on the number of points in a subinterval, these points are uniformly distributed in the subinterval. Making the parameters λ and p implicit, we write

$$MP_n(s) := \Pr_\lambda\{E^{(p)}(s)\} \quad \text{if} \quad [s] = n .$$

Clearly, $MP_0(s) = 0$. The auxiliary probabilities needed are directly analogous to those of Theorem 1, except that the number of points is not a fixed k but a random ν. With this single change, auxiliary probabilities $MP1_n(\underset{\sim}{s})$, $MP2_{nm}(\underset{\sim}{s})$, and $MAP_n(t)$ are defined corresponding to P1, P2, and AP above. The probability $MP_n(s)$, from which the generating function $m(\lambda,s)$ follows, is given by:

<u>Theorem 2.</u> The probabilities MP may be computed recursively from the following equations:

$$MP_n(s) = E\{MAP_n(\max_{1\le i\le \nu} X_i)\} = \lambda \int_0^{s-n} e^{-\lambda t} MAP_n(s-t)dt + e^{-\lambda(s-n)} MP_{n-1}(n) \tag{1}$$

$$MAP_n(t) = \sum_{\alpha=0}^{p-2} \phi_\lambda(1,\alpha) + \sum_{\alpha=0}^{p-2} \phi_\lambda(n-(t-1),\alpha) E\left\{MP2_{n,p-1-\alpha}\left((t-1,\underset{\sim}{X}_\alpha(t-1,n),\underset{\sim}{Z}_{p-1-\alpha}(n,t))\right)\right\} +$$
$$E\left\{MP1_{n-1}\left((t-1,\underset{\sim}{Z}_{p-1}(t-1,n))\right)\right\} \tag{2}$$

$$MP2_{nm}((t_0,t_1,\ldots,t_{p-1})) =$$

$$\begin{cases} MP1_{n-1}(t_0,t_1,\ldots,t_{p-1}) & m = 0 \\ \phi_\lambda(t_0-(t_{p-1}-1),0) + \sum_{\alpha=1}^{p-2} \phi_\lambda(t_0-(t_{p-1}-1),\alpha) E\left\{MP2_{n(m-\alpha)_+}\left((t_{p-1}-1,\underset{\sim}{X}_\alpha(t_{p-1}-1,t_0),\right.\right. & \\ \qquad\qquad \left.\left. t_1,t_2,\ldots,t_{p-1-\alpha})\right)\right\} + & \\ \qquad E\left\{MP1_{n-1}\left(t_{p-1}-1,\underset{\sim}{Z}_{p-1}(t_{p-1}-1,t_0)\right)\right\} & m > 0 \end{cases} \tag{3}$$

$$\mathrm{MP1}_n((t_0,t_1,\ldots,t_{p-1})) = \phi_\lambda(t_0-(t_{p-1}-1),0) +$$

$$\sum_{\alpha=0}^{p-2} \phi_\lambda(n-(t_{p-1}-1),\alpha)$$

$$\left[\sum_{\beta=(1-\alpha)_+}^{p-2-\alpha} \phi_\lambda(t_0-n,\beta)E\left\{\mathrm{MP2}_{n\beta}\left((t_{p-1}-1,\underset{\sim}{X}_\alpha(t_{p-1}-1,n),\underset{\sim}{X}'_\beta(n,t_0),t_1,\ldots,t_{p-1-\alpha-\beta})\right)\right\} +\right.$$

$$\left. + E\left\{\mathrm{MP2}_{n\,p-1-\alpha}\left((t_{p-1}-1,\underset{\sim}{X}_\alpha(t_{p-1}-1,n),\underset{\sim}{Z}_{p-1-\alpha}(n,t_0))\right)\right\}\right] +$$

$$+ E\left\{\mathrm{MP2}_{n-1}\left((t_{p-1}-1,Z_{p-1}(t_{p-1}-1,n))\right)\right\}. \tag{4}$$

Proof. In the above equations $\underset{\sim}{X}_\alpha(a,b)$, $\underset{\sim}{X}'_\alpha(a,b)$ represent random vectors of length α distributed as the order statistics of α points distributed randomly in the interval (a,b). $\underset{\sim}{Z}_\alpha(a,b)$ represents a random vector of length α distributed as the first α points of a Poisson process (rate λ) in (a,b). (Alternatively, z_α has a gamma distribution, and $(z_1,z_2,\ldots,z_{\alpha-1}) \sim \underset{\sim}{X}_{\alpha-1}(a,z_\alpha)$). The function $\phi_\lambda(t,k)$ is the Poisson probability $e^{-\lambda t}\frac{(t\lambda)^k}{k!}$ of k points in an interval of length t.

With the above definitions, Theorem 2 is derived exactly as Theorem 1.

As an example, we give here the explicit forms of the probablities $P\{$p-stretch of k points $\geq t\}$ as functions of k, for $p = 2,3,4,5$, valid for $t \geq \frac{1}{2}$. These formulas are obtained from the function MP_1 given in Theorem 2. (Values of k not covered by the formulas below are treated in the next section.)

$\underline{p = 2,\ k \geq 2}$

$$(k-1)(1-t)^k \tag{5}$$

$\underline{p = 3,\ k \geq 3}$

$$\{4-k+t(k(k-1)-4)\}(1-t)^{k-1} \tag{6}$$

$\underline{p = 4,\ k \geq 5}$

$$\{2-t(k^2-5k+4)+t^2(\frac{k(k-1)^2}{2} - 4k+2)\}(1-t)^{k-2} \tag{7}$$

$\underline{p = 5,\ k \geq 7}$

$$\{2+2t(k-3)-t^2\ \frac{k^3-7k^2+14k-12}{2} + t^3\ \frac{(k-1)^2(k^2-2k+12)}{6}\}\ (1-t)^{k-3} \tag{8}$$

4. Simple Results for Large Stretches.

For k points $Y_1 < Y_2 < \cdots < Y_k$, there is of course only one k-stretch, which is the range $Y_k - Y_1$. This random variable has a simple distribution, which can be shown by several standard methods to be: $\Pr\{Y_k - Y_1 \le t\} = kt^{k-1} - (k-1)t^k \quad 0 \le t \le 1$. We will derive generating functions here which give the distributions of the maximum p-stretch of k points, whenever $p \ge k/2 + 1$. The key simplification in these cases is that since the first p-stretch ends at Y_p and the last p-stretch begins at Y_{k+1-p}, no point Y_j can both begin and end a p-stretch $(k+1-p > p)$. In any of these cases, it will be shown that $\Pr\{\tilde{Z}^{(p)}(s) \ge 1\}$ can be given in terms of only two functions of s, for s in the range $(1,2)$ and $(2,\infty)$, respectively

For k points in $(0,s)$, we define the m-range to be the maximum $k+1-m$ stretch. Thus, *the 1-range is the usual range*, and the m-range is the largest of the m distances $Y_k - Y_m, Y_{k-1} - Y_{m-1}, \ldots, Y_{k-m+1} - Y_1$. We will consider a Poisson process on $(0,s)$ as above and find the probability $RP^{(m)}(s)$ that the m-range of this process is ≥ 1. This probability gives the generating function, as in section 2 above, of the probabilities $p_k(s) := \Pr\{\text{m-range of k points in } (0,s) \ge 1\}$, from which $\Pr\{\text{m-range of k points in } (0,1) \ge t\}$ may be derived for any t. We will assume that $k \ge 2m$ in what follows, and note that if $p \ge k/2 + 1$, then the maximum p-stretch of k points is the m-range, with $m = k+1-p$, so that in this case $2m = 2k+2-2p \le k$. Thus the generating functions for the m-range give the probabilities for the p-stretches when $p \ge k/2 + 1$.

Theorem 3. For $1 < s \leq 2$, the probabilities $RP^{(m)}$ may be computed from the following equations:

$$RP^{(m)}(s) = \int_0^{s-1} DNS_1(t) RP1^{(m)}(s-t)dt \qquad (1)$$

$$RP1^{(m)}(t) = PGE_m(t-1) + \sum_{\alpha=1}^{m-1} PRB_\alpha(t-1) MP2_{\alpha\, m-\alpha}(t-1,t) \qquad (2)$$

$$MP2_{jk}(u,v) = \int_0^{v-1} DNS_k(s) MP3_j(u,v-s) \qquad (3)$$

$$MP3_j(u,v) = \Pi(j,u,u-(v-1),0) + \sum_{\alpha=1}^{j-1} \Pi(j,u,u-(v-1),\alpha) MP2_{j-\alpha\ \alpha}(v-1,v) \qquad (4)$$

Proof. In the above equations,

$$PRB_\alpha(t) = e^{-\lambda t} \frac{(\lambda t)^\alpha}{\alpha!}, \quad PGE_j(t) = 1 - \sum_{\alpha=0}^{j-1} PRB_\alpha(t),$$

and

$$DNS_j(t) = \frac{d\ PGE_j(t)}{dt}$$

are the probability of α points, the probability of greater than or equal to j points, and the density of the hitting time of the j-th point, respectively, for a Poisson process. Let $E^{(m)}(s)$ be the event that the m-range of the points in $(0,s) \geq 1$.

Let $E_1^{(m)}(t) = \{E^{(m)}(t)$ and a point of the process occurs at $t\}$, and $RP1^{(m)}(t) = \Pr\{E_1^{(m)}(t)\}$. Since the density of the distance t of the rightmost point in $(0,s)$ from s is $DNS_1(t)$, equation (1) follows. To calculate the probability of $E_1^{(m)}(t)$ we observe that if there are at least m points in $(0,t-1)$ then $E_1^{(m)}(t)$ occurs, since in that case, the m-th point from the left $Y_m < t-1$, so that $Y_k - Y_m = t - Y_m > 1$. The probability of this is $PGE_m(t-1)$. If there are $1 \leq \alpha < m$ points in $(0,t-1)$, define

$$E2_{\alpha\beta}(u,v) = \{E_1^{(\alpha)}(y_\beta) \text{ and } \alpha \text{ points in } (0,u)\},$$

where y_β is the β-th point to the left of v. Letting $MP2_{\alpha\beta} = \Pr\{E2_{\alpha\beta}\}$ equation (2) follows. To derive (3), note that the density of $v-y_\beta$ is DNS_β, let

$$E3_j(u,v) = \{E_1^{(j)}(v) \text{ and } j \text{ points in } (0,u)\} ,$$

and define $MP3_j(u,v) = \Pr\{E3_j(u,v)\}$. Letting N denote the number of points in $(0,u)$ which fall in the subinterval $(v-1,u)$, $E3_j(u,v)$ may be written as the disjoint union:

$$E3_j(u,v) = \{N=0\} + \bigcup_{\alpha=1}^{j-1} \{E2_{j-\alpha\,\alpha}(v-1,v) \cap \{N=\alpha\}\}$$

Since $\Pr\{N=\alpha\} = \Pi(j,u,u-(v-1),\alpha)$, equation (4) follows and the theorem is proved.

In proving Theorem 3, we assumed that $1 < s < 2$, specifically in setting the upper limit of the integral in equation (1). For the case that $s > 2$, an analogous theorem, proved in exactly the same fashion, holds. We denote the probability that the m-range exceeds 1 in this case by $R2P^{(m)}(s)$, to distinguish it from the previous result, which will be used explicitly.

<u>Theorem 4</u>. For $s \geq 2$, the probabilities $RP2^{(m)}(s)$ may be computed from the following equations:

$$R2P^{(m)}(s) = \int_0^{s-2} R2P1^{(m)}(s-t) \times DNS_1(t) + PRB_0(s-2)RP^{(m)}(2) \tag{5}$$

$$R2P1^{(m)}(t) = PGE_m(t-1) + \sum_{\alpha=1}^{m-1} PRB_\alpha(t-1)M2P2_{\alpha\, m-\alpha}(t-1,t-1,t) \tag{6}$$

$$M2P2_{jk}(u,v,w) = \int_0^{v-w} DNS_k(s)M2P3_j(u,w,v-s) \tag{7}$$

$$M2P3_j(u,w,v) = \Pi(j,u,u-(v-1),0) + \sum_{\alpha=1}^{j-1} \Pi(j,u,u-(v-1),\alpha)M2P2_{j-\alpha\,\alpha}(v-1,w,v) \tag{8}$$

From theorems 3 and 4, for example, we obtain:

$$\Pr\{\text{maximum } k\text{-1 stretch of } k \text{ points} \leq t\} = 1-2k(t^{k-1}-t^k) - (2t-1)_+^k \tag{9}$$

Pr{maximum k-2 stretch of k points $\leq$ t} $(k \geq 5)$

$$1-8(2t-1)_+^k + (5k(t-1)+7t)t^{k-1} - \frac{k(t-1)((k-17)t-k+9)}{2}(2t-1)_+^{k-2} \qquad (10)$$

5. Tables.

In tables 1-4 which follow, the main results for the uniform distribution are given. Table 1 gives the cumulative distribution function for the maximum p-stretch. For example, under " 2 stretches" in Table 1, we find the entry:

POINTS = 4

$$(\tfrac{1}{2}, 1) \quad 3t^4 - 12t^3 + 18t^2 - 12t + 3$$

$$(\tfrac{1}{3}, \tfrac{1}{2}) \quad -45t^4 + 84t^3 - 54t^2 + 12t$$

$$(\tfrac{1}{4}, \tfrac{1}{3}) \quad 36t^4 - 24t^3 + 1 \ .$$

This entry gives $\Pr\{\text{max 2-stretch of 4 points} \le t\}$ for t in each of the intervals indicated at the left. For any fixed number of points, the last form given also holds for all smaller values of t.

Table 2 gives the quantiles of the maximum p-stretch, derived from the exact formulas in Table 1, along with the approximate quantiles obtained by treating the stretches as independent (marginal Beta distributions.) Since this approximation is exact for the maximum p-stretch of p points, these cases are omitted from the table.

Table 3 gives exact formulas for the A^{th} moment of the maximum p-stretch, $E\{\tilde{Z}^{(p)A}\}$, in a somewhat compact form. That is, the desired moment (A^{th} moment of the maximum p-stretch of k points) always involves a factor $\prod_{i=0}^{k} \frac{1}{A+i}$, and this factor is omitted from the tables. Thus, under "3-stretch, 4 pts.", for example, the entry $\frac{24}{2^A} + 48A$ indicates that the A^{th} moment is $\frac{24/2^A + 48A}{A(A+1)(A+2)(A+3)(A+4)}$. Numerical values of the moments for $A = 1,2,3,4,5$ are given in Table 4.

Table 1. Exact Distribution of the Maximum p-Stretch.

$\boxed{\begin{matrix}2\\ \text{STRETCHES}\end{matrix}}$ (Gaps)

POINTS = 2

(1 / 2 , 1 / 1)	$T^{2} - 2T + 1$

POINTS = 3

(1 / 2 , 1 / 1)	$-2T^{3} + 6T^{2} - 6T + 2$
(1 / 3 , 1 / 2)	$6T^{3} - 6T^{2} + 1$

POINTS = 4

(1 / 2 , 1 / 1)	$3T^{4} - 12T^{3} + 18T^{2} - 12T + 3$
(1 / 3 , 1 / 2)	$-45T^{4} + 84T^{3} - 54T^{2} + 12T$
(1 / 4 , 1 / 3)	$36T^{4} - 24T^{3} + 1$

POINTS = 5

(1 / 2 , 1 / 1)	$-4T^{5} + 20T^{4} - 40T^{3} + 40T^{2} - 20T + 4$
(1 / 3 , 1 / 2)	$188T^{5} - 460T^{4} + 440T^{3} - 200T^{2} + 40T - 2$
(1 / 4 , 1 / 3)	$-784T^{5} + 1160T^{4} - 640T^{3} + 160T^{2} - 20T + 2$
(1 / 5 , 1 / 4)	$240T^{5} - 120T^{4} + 1$

POINTS = 6

(1 / 2 , 1 / 1)	$5T^{6} - 30T^{5} + 75T^{4} - 100T^{3} + 75T^{2} - 30T + 5$
(1 / 3 , 1 / 2)	$-635T^{6} + 1890T^{5} - 2325T^{4} + 1500T^{3} - 525T^{2} + 90T - 5$
(1 / 4 , 1 / 3)	$6655T^{6} - 12690T^{5} + 9825T^{4} - 3900T^{3} + 825T^{2} - 90T + 5$
(1 / 5 , 1 / 4)	$-13825T^{6} + 18030T^{5} - 9375T^{4} + 2500T^{3} - 375T^{2} + 30T$
(1 / 6 , 1 / 5)	$1800T^{6} - 720T^{5} + 1$

POINTS = 7

(1 / 2 , 1 / 1) $-6T^7 + 42T^6 - 126T^5 + 210T^4 - 210T^3 + 126T^2 - 42T + 6$

(1 / 3 , 1 / 2) $1914T^7 - 6678T^6 + 9954T^5 - 8190T^4 + 3990T^3 - 1134T^2 + 168T - 9$

(1 / 4 , 1 / 3) $-41826T^7 + 95382T^6 - 92106T^5 + 48510T^4 - 14910T^3 + 2646T^2 - 252T + 11$

(1 / 5 , 1 / 4) $203934T^7 - 334698T^6 + 230454T^5 - 85890T^4 + 18690T^3 - 2394T^2 + 168T - 4$

(1 / 6 , 1 / 5) $-264816T^7 + 321552T^6 - 163296T^5 + 45360T^4 - 7560T^3 + 756T^2 - 42T + 2$

(1 / 7 , 1 / 6) $15120T^7 - 5040T^6 + 1$

POINTS = 8

(1 / 2 , 1 / 1) $7T^8 - 56T^7 + 196T^6 - 392T^5 + 490T^4 - 392T^3 + 196T^2 - 56T + 7$

(1 / 3 , 1 / 2) $-5369T^8 + 21448T^7 - 37436T^6 + 37240T^5 - 23030T^4 + 9016T^3 - 2156T^2 + 280T - 14$

(1 / 4 , 1 / 3) $224266T^8 - 590912T^7 + 676984T^6 - 439040T^5 + 175420T^4 - 43904T^3 + 6664T^2 - 560T + 21$

(1 / 5 , 1 / 4) $-2069494T^8 + 3996608T^7 - 3337096T^6 + 1568000T^5 - 451780T^4 + 81536T^3 - 9016T^2 + 560T - 14$

(1 / 6 , 1 / 5) $6133631T^8 - 9128392T^7 + 5850404T^6 - 2107000T^5 + 466970T^4 - 65464T^3 + 5684T^2 - 280T + 7$

(1 / 7 , 1 / 6) $-5623681T^8 + 6548024T^7 - 3294172T^6 + 941192T^5 - 168070T^4 + 19208T^3 - 1372T^2 + 56T$

(1 / 8 , 1 / 7) $141120T^8 - 40320T^7 + 1$

POINTS = 9

$(1/2, 1/1)$ $-8T^9 + 72T^8 - 288T^7 + 672T^6 - 1008T^5 + 1008T^4 - 672T^3 + 288T^2 - 72T + 8$

$(1/3, 1/2)$ $14328T^9 - 64440T^8 + 128736T^7 - 149856T^6 + 111888T^5 - 55440T^4 + 18144T^3 - 3744T^2 + 432T - 20$

$(1/4, 1/3)$ $-1087920T^9 + 3242304T^8 - 4280256T^7 + 3279360T^6 - 1602720T^5 + 516096T^4 - 108864T^3 + 14400T^2 - 1080T + 36$

$(1/5, 1/4)$ $17262160T^9 - 38045376T^8 + 37007424T^7 - 20805120T^6 + 7428960T^5 - 1741824T^4 + 267456T^3 - 25920T^2 + 1440T - 34$

$(1/6, 1/5)$ $-92112840T^9 + 158829624T^8 - 120492576T^7 + 52694880T^6 - 14621040T^5 + 2668176T^4 - 320544T^3 + 24480T^2 - 1080T + 22$

$(1/7, 1/6)$ $190062648T^9 - 264433608T^8 + 161682912T^7 - 57040032T^6 + 12812688T^5 - 1904112T^4 + 187488T^3 - 11808T^2 + 432T - 6$

$(1/8, 1/7)$ $-132766208T^9 + 150632064T^8 - 75497472T^7 + 22020096T^6 - 4128768T^5 + 516096T^4 - 43008T^3 + 2304T^2 - 72T + 2$

$(1/9, 1/8)$ $1451520T^9 - 362880T^8 + 1$

POINTS = 10

$$(1/2,\ 1/1)\quad 9T^{10} - 90T^{9} + 405T^{8} - 1080T^{7} + 1890T^{6} - 2268T^{5}$$
$$+ 1890T^{4} - 1080T^{3} + 405T^{2} - 90T + 9$$

$$(1/3,\ 1/2)\quad -36855T^{10} + 184230T^{9} - 414315T^{8} + 551880T^{7} - 461950T^{6}$$
$$+ 288036T^{5} - 119070T^{4} + 33480T^{3} - 6075T^{2} + 630T - 27$$

$$(1/4,\ 1/3)\quad 4923261T^{10} - 16349490T^{9} + 24386265T^{8} - 21493080T^{7}$$
$$+ 12377610T^{6} - 4855788T^{5} + 1309770T^{4} - 238680T^{3} + 27945T^{2} - 1890T + 57$$

$$(1/5,\ 1/4)\quad -127197315T^{10} + 313951950T^{9} - 347202855T^{8} + 226233000T^{7}$$
$$- 96002550T^{6} + 27658260T^{5} - 5463990T^{4} + 729000T^{3} - 62775T^{2} + 3150T - 69$$

$$(1/6,\ 1/5)\quad 1103271435T^{10} - 2146985550T^{9} + 1867640695T^{8}$$
$$- 955017000T^{7} + 317434950T^{6} - 71566740T^{5} + 11073510T^{4} - 1161000T^{3}$$
$$+ 78975T^{2} - 3150T + 57$$

$$(1/7,\ 1/6)\quad -3975887349T^{10} + 6318279090T^{9} - 4481307585T^{8}$$
$$+ 1866737880T^{7} - 505576890T^{6} + 93035628T^{5} - 11787930T^{4} + 1016280T^{3}$$
$$- 57105T^{2} + 1890T - 27$$

$$(1/8,\ 1/7)\quad 6193221615T^{10} - 8209019430T^{9} + 4857670035T^{8}$$
$$- 1690967880T^{7} + 383849550T^{6} - 59437476T^{5} + 6363630T^{4} - 465480T^{3}$$
$$+ 22275T^{2} - 630T + 9$$

$$(1/9,\ 1/8)\quad -3470454801T^{10} + 3870576090T^{9} - 1937102445T^{8}$$
$$+ 573956280T^{7} - 111602610T^{6} + 14880348T^{5} - 1377810T^{4} + 87480T^{3} - 3645T^{2}$$
$$+ 90T$$

$$(1/10,\ 1/9)\quad 16329600T^{10} - 3628800T^{9} + 1$$

3

STRETCHES

POINTS = 3

$(1/2, 1/1)$ $2T^{3} - 3T^{2} + 1$

POINTS = 4

$(1/2, 1/1)$ $-8T^{4} + 24T^{3} - 24T^{2} + 8T$

$(1/3, 1/2)$ $8T^{4} - 8T^{3} + 1$

POINTS = 5

$(1/2, 1/1)$ $16T^{5} - 65T^{4} + 100T^{3} - 70T^{2} + 20T - 1$

$(1/3, 1/2)$ $32T^{5} - 25T^{4} + 1$

POINTS = 6

$(1/2, 1/1)$ $-26T^{6} + 132T^{5} - 270T^{4} + 280T^{3} - 150T^{2} + 36T - 2$

$(1/3, 1/2)$ $-1306T^{6} + 2820T^{5} - 2430T^{4} + 1080T^{3} - 270T^{2} + 36T - 1$

$(1/4, 1/3)$ $152T^{6} - 96T^{5} + 1$

POINTS = 7

$(1/2, 1/1)$ $38T^{7} - 231T^{6} + 588T^{5} - 805T^{4} + 630T^{3} - 273T^{2} + 56T$

$- 3$

$(1/3, 1/2)$ $8902T^{7} - 23863T^{6} + 26460T^{5} - 15645T^{4} + 5320T^{3} - 1050T^{2}$

$+ 112T - 4$

$(1/4, 1/3)$ $802T^{7} - 427T^{6} + 1$

POINTS = 8

(1 / 2 , 1 / 1) $-52T^{8} + 368T^{7} - 1120T^{6} + 1904T^{5} - 1960T^{4} + 1232T^{3}$
$- 448T^{2} + 80T - 4$

(1 / 3 , 1 / 2) $-42036T^{8} + 133488T^{7} - 180320T^{6} + 134512T^{5} - 60200T^{4}$
$+ 16464T^{3} - 2688T^{2} + 240T - 8$

(1 / 4 , 1 / 3) $-322944T^{8} + 653184T^{7} - 573440T^{6} + 286720T^{5} - 89600T^{4}$
$+ 17920T^{3} - 2240T^{2} + 160T - 4$

(1 / 5 , 1 / 4) $4736T^{8} - 2176T^{7} + 1$

POINTS = 9

(1 / 2 , 1 / 1) $68T^{9} - 549T^{8} + 1944T^{7} - 3948T^{6} + 5040T^{5} - 4158T^{4}$
$+ 2184T^{3} - 684T^{2} + 108T - 5$

(1 / 3 , 1 / 2) $165444T^{9} - 606501T^{8} + 966168T^{7} - 873516T^{6} + 490896T^{5}$
$- 176526T^{4} + 40320T^{3} - 5616T^{2} + 432T - 13$

(1 / 4 , 1 / 3) $4653168T^{9} - 10762929T^{8} + 10886400T^{7} - 6323520T^{6}$
$+ 2327976T^{5} - 564354T^{4} + 90216T^{3} - 9180T^{2} + 540T - 13$

(1 / 58, 1 / 4) $30832T^{9} - 12465T^{8} + 1$

POINTS = 10

$(1/2, 1/1)\quad -86T^{10} + 780T^{9} - 3150T^{8} + 7440T^{7} - 11340T^{6} + 11592T^{5}$

$\qquad - 7980T^{4} + 3600T^{3} - 990T^{2} + 140T - 6$

$(1/3, 1/2)\quad -581718T^{10} + 2417420T^{9} - 4438350T^{8} + 4722960T^{7}$

$\qquad - 3210060T^{6} + 1446984T^{5} - 434700T^{4} + 85200T^{3} - 10350T^{2} + 700T - 19$

$(1/4, 1/3)\quad -43044510T^{10} + 112467260T^{9} - 129972150T^{8} + 87381840T^{7}$

$\qquad - 37819740T^{6} + 11008872T^{5} - 2183580T^{4} + 291600T^{3} - 25110T^{2} + 1260T - 27$

$(1/5, 1/4)\quad -136498846T^{10} + 273358140T^{9} - 246093750T^{8} + 131250000T^{7}$

$\qquad - 45937500T^{6} + 11025000T^{5} - 1837500T^{4} + 210000T^{3} - 15750T^{2} + 700T - 13$

$(1/6, 1/5)\quad 219904T^{10} - 79360T^{9} + 1$

4 STRETCHES

POINTS = 4

(1 / 2 , 1 / 1) $3 T^{4} - 4 T^{3} + 1$

POINTS = 5

(1 / 2 , 1 / 1) $- 22 T^{5} + 70 T^{4} - 80 T^{3} + 40 T^{2} - 10 T + 2$

(1 / 3 , 1 / 2) $10 T^{5} - 10 T^{4} + 1$

POINTS = 6

(1 / 2 , 1 / 1) $53 T^{6} - 222 T^{5} + 360 T^{4} - 280 T^{3} + 105 T^{2} - 18 T + 2$

(1 / 3 , 1 / 2) $37 T^{6} - 30 T^{5} + 1$

POINTS = 7

(1 / 2 , 1 / 1) $- 100 T^{7} + 518 T^{6} - 1092 T^{5} + 1190 T^{4} - 700 T^{3} + 210 T^{2} - 28 T + 2$

(1 / 3 , 1 / 2) $140 T^{7} - 98 T^{6} + 1$

POINTS = 8

(1 / 2 , 1 / 1) $166 T^{8} - 1024 T^{7} + 2660 T^{6} - 3752 T^{5} + 3080 T^{4} - 1456 T^{3} + 364 T^{2} - 40 T + 2$

(1 / 3 , 1 / 2) $- 32186 T^{8} + 87104 T^{7} - 102060 T^{6} + 68040 T^{5} - 28350 T^{4} + 7560 T^{3} - 1260 T^{2} + 120 T - 4$

(1 / 4 , 1 / 3) $619 T^{8} - 376 T^{7} + 1$

POINTS = 9

(1 / 2 , 1 / 1) $- 254 T^{9} + 1818 T^{8} - 5616 T^{7} + 9744 T^{6} - 10332 T^{5} + 6804 T^{4} - 2688 T^{3} + 576 T^{2} - 54 T + 2$

(1 / 3 , 1 / 2) $234882 T^{9} - 701478 T^{8} + 898128 T^{7} - 644112 T^{6} + 283500 T^{5} - 78624 T^{4} + 13440 T^{3} - 1296 T^{2} + 54 T + 1$

(1 / 4 , 1 / 3) $3060 T^{9} - 1638 T^{8} + 1$

POINTS = 10

(1 / 2 , 1 / 1) $367 T^{10} - 2990 T^{9} + 10710 T^{8} - 22080 T^{7} + 28770 T^{6} - 24444 T^{5} + 13440 T^{4} - 4560 T^{3} + 855 T^{2} - 70 T + 2$

(1 / 3 , 1 / 2) $- 1304337 T^{10} + 4443090 T^{9} - 6571530 T^{8} + 5523840 T^{7} - 2900310 T^{6} + 984060 T^{5} - 215250 T^{4} + 29160 T^{3} - 2205 T^{2} + 70 T + 1$

(1 / 4 , 1 / 3) $16368 T^{10} - 7860 T^{9} + 1$

$$\boxed{\begin{matrix}5\\ \text{STRETCHES}\end{matrix}}$$

POINTS = 5

(1 / 2 , 1 / 1) $4 T^{5} - 5 T^{4} + 1$

POINTS = 6

(1 / 2 , 1 / 1) $- 52 T^{6} + 180 T^{5} - 240 T^{4} + 160 T^{3} - 60 T^{2} + 12 T$

(1 / 3 , 1 / 2) $12 T^{6} - 12 T^{5} + 1$

POINTS = 7

(1 / 2 , 1 / 1) $138 T^{7} - 595 T^{6} + 1008 T^{5} - 840 T^{4} + 350 T^{3} - 63 T^{2} + 2$

(1 / 3 , 1 / 2) $42 T^{7} - 35 T^{6} + 1$

POINTS = 8

(1 / 2 , 1 / 1) $- 294 T^{8} + 1552 T^{7} - 3360 T^{6} + 3808 T^{5} - 2380 T^{4} + 784 T^{3}$

$- 112 T^{2} + 2$

(1 / 3 , 1 / 2) $154 T^{8} - 112 T^{7} + 1$

POINTS = 9

$(1/2, 1/1)$ $544 T^{9} - 3402 T^{8} + 9000 T^{7} - 13020 T^{6} + 11088 T^{5} - 5544 T^{4}$
$+ 1512 T^{3} - 180 T^{2} + 2$

$(1/3, 1/2)$ $576 T^{9} - 378 T^{8} + 1$

POINTS = 10

$(1/2, 1/1)$ $- 918 T^{10} + 6640 T^{9} - 20790 T^{8} + 36720 T^{7} - 39900 T^{6}$
$+ 27216 T^{5} - 11340 T^{4} + 2640 T^{3} - 270 T^{2} + 2$

$(1/3, 1/2)$ $- 824214 T^{10} + 2754160 T^{9} - 4133430 T^{8} + 3674160 T^{7}$
$- 2143260 T^{6} + 857304 T^{5} - 238140 T^{4} + 45360 T^{3} - 5670 T^{2} + 420 T - 13$

$(1/4, 1/3)$ $2472 T^{10} - 1460 T^{9} + 1$

6 STRETCHES

POINTS = 6

(1 / 2 , 1 / 1) $5T^{6} - 6T^{5} + 1$

POINTS = 7

(1 / 2 , 1 / 1) $-114T^{7} + 434T^{6} - 672T^{5} + 560T^{4} - 280T^{3} + 84T^{2} - 14T + 2$

(1 / 3 , 1 / 2) $14T^{7} - 14T^{6} + 1$

POINTS = 8

(1 / 2 , 1 / 1) $303T^{8} - 1320T^{7} + 2240T^{6} - 1792T^{5} + 560T^{4} + 112T^{3} - 140T^{2} + 40T - 3$

(1 / 3 , 1 / 2) $47T^{8} - 40T^{7} + 1$

POINTS = 9

(1 / 2 , 1 / 1) $-728T^{9} + 3906T^{8} - 8640T^{7} + 10080T^{6} - 6552T^{5} + 2268T^{4} - 336T^{3} + 2$

(1 / 3 , 1 / 2) $168T^{9} - 126T^{8} + 1$

POINTS = 10

(1 / 2 , 1 / 1) $1482T^{10} - 9380T^{9} + 25200T^{8} - 37200T^{7} + 32550T^{6} - 16884T^{5} + 4830T^{4} - 600T^{3} + 2$

(1 / 3 , 1 / 2) $618T^{10} - 420T^{9} + 1$

7 STRETCHES

POINTS = 7

(1 / 2 , 1 / 1) $6T^{7} - 7T^{6} + 1$

POINTS = 8

(1 / 2 , 1 / 1) $-240T^{8} + 1008T^{7} - 1792T^{6} + 1792T^{5} - 1120T^{4} + 448T^{3} - 112T^{2} + 16T$

(1 / 3 , 1 / 2) $16T^{8} - 16T^{7} + 1$

POINTS = 9

(1 / 2 , 1 / 1) $564T^{9} - 2349T^{8} + 3456T^{7} - 1344T^{6} - 2016T^{5} + 3024T^{4} - 1848T^{3} + 612T^{2} - 108T + 9$

(1 / 3 , 1 / 2) $52T^{9} - 45T^{8} + 1$

POINTS = 10

(1 / 2 , 1 / 1) $-1610T^{10} + 8820T^{9} - 20160T^{8} + 24960T^{7} - 18480T^{6} + 9072T^{5} - 3780T^{4} + 1680T^{3} - 630T^{2} + 140T - 12$

(1 / 3 , 1 / 2) $182T^{10} - 140T^{9} + 1$

$$\boxed{\begin{array}{c}8\\ \text{STRETCHES}\end{array}}$$

POINTS = 8

(1 / 2 , 1 / 1) $7 T^{8} - 8 T^{7} + 1$

POINTS = 9

(1 / 2 , 1 / 1) $- 494 T^{9} + 2286 T^{8} - 4608 T^{7} + 5376 T^{6} - 4032 T^{5} + 2016 T^{4}$

$- 672 T^{3} + 144 T^{2} - 18 T + 2$

(1 / 3 , 1 / 2) $18 T^{9} - 18 T^{8} + 1$

POINTS = 10

(1 / 2 , 1 / 1) $825 T^{10} - 2610 T^{9} + 11520 T^{7} - 23520 T^{6} + 24192 T^{5}$

$- 15120 T^{4} + 6000 T^{3} - 1485 T^{2} + 210 T - 12$

(1 / 3 , 1 / 2) $57 T^{10} - 50 T^{9} + 1$

$$\boxed{\begin{array}{c} 9 \\ \text{STRETCHES} \end{array}}$$

POINTS = 9

$$(1/2, 1/1) \quad 8T^{9} - 9T^{8} + 1$$

POINTS = 10

$$(1/2, 1/1) \quad -1004T^{10} + 5100T^{9} - 11520T^{8} + 15360T^{7} - 13440T^{6}$$
$$+ 8064T^{5} - 3360T^{4} + 960T^{3} - 180T^{2} + 20T$$
$$(1/3, 1/2) \quad 20T^{10} - 20T^{9} + 1$$

$$\boxed{\begin{array}{c} 10 \\ \text{STRETCHES} \end{array}}$$

POINTS = 10

$$(1/2, 1/1) \quad 9T^{10} - 10T^{9} + 1$$

Table 2. Quantiles of the Maximum p-Stretch.

$\boxed{\begin{matrix}2\\ \text{STRETCH}\end{matrix}}$

2 POINTS

P	QUANTILE
0.05	0.0253205763
0.1	0.051316784
0.25	0.133974597
0.5	0.292892978
0.75	0.499979568
0.9	0.68377261
0.95	0.77639345
0.99	0.90000014
0.995	0.929289386
0.999	0.96837668
0.9995	0.977638766

3 POINTS

P	QUANTILE	BETA APPROX
0.05	0.096012638	0.080905483
0.1	0.139142558	0.119012401
0.25	0.233089015	0.206301257
0.5	0.36117606	0.335897967
0.75	0.499999568	0.488313243
0.9	0.63159709	0.62839846
0.95	0.707598[illegible]	0.706365146
0.99	0.82930195	0.82889323
0.995	0.864279315	0.864276454
0.999	0.92063002	0.9207807
0.9995	0.93[illegible]0027	0.937254466

4 POINTS

P	QUANTILE	BETA APPROX
0.05	0.137981936	0.108524845
0.1	0.178533124	0.14442496
0.25	0.25685267	0.220060872
0.5	0.354245707	0.326057956
0.75	0.462665126	0.45010619
0.9	0.57271341	0.569001[illegible]2
0.95	0.64069609	0.63918642
0.99	0.75971846	0.75958971
0.995	0.797948405	0.79798464
0.999	0.86487917	0.86527399
0.9995	0.88637785	0.867059[illegible]

5 POINTS

P	QUANTILE	BETA APPROX
0.05	0.156986758	0.12020258[illegible]
0.1	0.191742465	0.15233092
0.25	0.[illegible]554774[illegible]	0.21[illegible]387
0.5	0.[illegible]4406	0.307637736
0.75	0.425447986	0.413511798
0.9	0.52182315	0.51809077
0.95	0.58372358	0.58216052
0.99	0.[illegible]829135	0.698184535
0.995	0.73734717	0.7374578
0.999	0.80963378	0.81033336
0.9995	0.83427577	0.83561[illegible]

6 POINTS

P	QUANTILE	BETA APPROX
0.05	0.163698718	0.12437[illegible]3
0.1	0.193106221	0.153074788
0.25	0.246035141	0.210516497
0.5	0.31254249[illegible]	0.288764518
0.75	0.39264159	0.38163714
0.9	0.47899966	0.47539095[illegible]
0.95	0.53584151	0.53430514
0.99	0.645045795	0.64498572
0.995	0.683771655	0.68398242
0.999	0.75817256	0.7593[illegible]6
0.9995	0.784555	0.78675512

7 POINTS

P	QUANTILE	BETA APPROX
0.05	0.164121198	0.124892758
0.1	0.189214276	0.150712535
0.25	0.234422252	0.201878116
0.5	0.292644065	0.27130942
0.75	0.364334626	0.354276225
0.9	0.44283919	0.439404055
0.95	0.49536852	0.493890326
0.99	0.59902148	0.599020526
0.995	0.63682417	0.637[illegible]0325
0.999	0.71142058	0.7131619[illegible]
0.9995	0.738627	0.7[illegible]165995

8 POINTS

P	QUANTILE	BETA APPROX
0.05	0.161369845	0.123561428
0.1	0.1831[illegible]	0.1469[illegible]44
0.25	0.22[illegible]08272	0.193126246
0.5	0.274[illegible]3193	0.255618617
0.75	0.339904353	0.330725238
0.9	0.412005945	0.408[illegible]
0.95	0.460819766	0.45941[illegible]
0.99	0.55[illegible]683	0.55914739[illegible]
0.995	0.59567122	0.59612[illegible]
0.999	0.66735401	0.67[illegible]
0.9995	0.69679503	0.70124[illegible]4

9 POINTS

P	QUANTILE	BETA APPROX
0.05	0.157081172	0.121287868
0.1	0.176251933	0.142718839
0.25	0.211614178	0.184742495
0.5	0.258919284	0.241614863
0.75	0.318691775	0.31030240.
0.9	0.38543[illegible]65	0.3823895[illegible]
0.95	0.43101839[illegible]	0.42969469[illegible]
0.99	0.52418856	0.5243[illegible]8
0.995	0.5594[illegible]83	0.56005911
0.999	0.63159613	0.63461641
0.9995	0.6589046	0.66471629

10 POINTS

P	QUANTILE	BETA APPROX
0.05	0.152129695	0.118551778
0.1	0.169294879	0.138295695
0.25	0.201359317	0.176892802
0.5	0.244805858	0.229113147
0.75	0.300131366	0.292442843
0.9	0.362302348	0.35945086
0.95	0.40505843	0.40381384
0.99	0.49350409	0.493[illegible]32974
0.995	0.527422465	0.[illegible]2817205
0.999	0.597675845	0.60141329
0.9995	0.62461714	0.63194422

3

STRETCH

3 POINTS

P	QUANTILE
0.05	0.13535075
0.1	0.19580035
0.25	0.326351687
0.5	0.499999568
0.75	0.6733464
0.9	0.80419973
0.95	0.86464933
0.99	0.94109683
0.995	0.95859961
0.999	0.98162989
0.9995	0.987033404

4 POINTS

P	QUANTILE	BETA APPROX
0.05	0.198283717	0.227026507
0.1	0.256137416	0.281723544
0.25	0.366807505	0.38572745
0.5	0.499999568	0.51324992
0.75	0.63319258	0.642282054
0.9	0.74386267	0.749050654
0.95	0.801716365	0.80500369
0.99	0.88792471	0.889019534
0.995	0.91182952	0.91250853
0.999	0.94912199	0.94934611
0.9995	0.959767856	0.95990614

5 POINTS

P	QUANTILE	BETA APPROX
0.05	0.230824038	0.25057654
0.1	0.281158015	0.29630331
0.25	0.37162077	0.381285235
0.5	0.47543387	0.48538737
0.75	0.583941974	0.594363734
0.9	0.68322997	0.69052081
0.95	0.739326906	0.74449973
0.99	0.83113436	0.83332209
0.995	0.859272525	0.86077265
0.999	0.907255674	0.90788512
0.9995	0.92235522	0.92279165

6 POINTS

P	QUANTILE	BETA APPROX
0.05	0.242950961	0.252873942
0.1	0.28564982	0.291984126
0.25	0.36015463	0.36431174
0.5	0.44574694	0.45364718
0.75	0.539066836	0.54970888
0.9	0.62975745	0.638154544
0.95	0.68362288	0.689992465
0.99	0.77718023	0.78029017
0.995	0.80767015	0.80993875
0.999	0.86260943	0.86370902
0.9995	0.88094477	0.88175348

7 POINTS

P	QUANTILE	BETA APPROX
0.05	0.245140597	0.247689769
0.1	0.281423137	0.28190379
0.25	0.343864962	0.34516482
0.5	0.417178676	0.423981234
0.75	0.499999568	0.510469005
0.9	0.58357005	0.59251933
0.95	0.63495116	0.64205222
0.99	0.72794108	0.73175959
0.995	0.75953535	0.76243738
0.999	0.818646	0.82019381
0.9995	0.8391662	0.84035449

8 POINTS

P	QUANTILE	BETA APPROX
0.05	0.241949603	0.23978953
0.1	0.273098513	0.27024674
0.25	0.326656863	0.32663016
0.5	0.39123492	0.397416636
0.75	0.46619086	0.47631674
0.9	0.5436769	0.55284648
0.95	0.59253363	0.6000524
0.99	0.68366961	0.68800406
0.995	0.715582415	0.71897365
0.999	0.77694468	0.77888064
0.9995	0.79286012	0.80038409

9 POINTS

P	QUANTILE	BETA APPROX
0.05	0.23605113	0.231051013
0.1	0.26312306	0.258528277
0.25	0.31004481	0.309482142
0.5	0.36806731	0.373867556
0.75	0.435790988	0.446523234
0.9	0.50901942	0.51820807
0.95	0.555432834	0.5631576
0.99	0.64403777	0.64873461
0.995	0.67578368	0.679545924
0.999	0.73811011	0.740405604
0.9995	0.76085429	0.762670085

10 POINTS

P	QUANTILE	BETA APPROX
0.05	0.22885661	0.22227244
0.1	0.252705142	0.247324511
0.25	0.294559047	0.293860957
0.5	0.34744315	0.35298972
0.75	0.41103892	0.42037825
0.9	0.47869067	0.48777537
0.95	0.52279429	0.5305858
0.99	0.60854964	0.61349253
0.995	0.639832065	0.643851794
0.999	0.70226626	0.7047811
0.9995	0.72543577	0.72749476

4

STRETCH

4 POINTS

P	QUANTILE
0.05	0.248604344
0.1	0.32046084
0.25	0.456321284
0.5	0.61427264
0.75	0.7569776
0.9	0.85744052
0.95	0.90238814
0.99	0.95600166
0.995	0.970554366
0.999	0.9867[illegible]4
0.9995	0.99081472

5 POINTS

P	QUANTILE	BETA APPROX
0.05	0.269850485	0.342477366
0.1	0.352529094	0.399298236
0.25	0.464922473	0.499999568
0.5	0.5913835	0.61438517
0.75	0.70999579	0.72303252
0.9	0.80236392	0.80807468
0.95	0.84876113	0.852687396
0.99	0.915864505	0.91709856
0.995	0.934035414	0.93484072
0.999	0.96220259	0.96244664
0.9995	0.97016387	0.970316455

6 POINTS

P	QUANTILE	BETA APPROX
0.05	0.30584483	0.357642695
0.1	0.359269664	0.4040894[illegible]6
0.25	0.451722258	0.48616366
0.5	0.55537467	0.581129596
0.75	0.658600375	0.675717875
0.9	0.74603324	0.75602201
0.95	0.793366	0.800024554
0.99	0.8683391	0.87097983
0.995	0.89078859	0.892577655
0.999	0.92856936	0.9293[illegible]
0.9995	0.94033675	0.94084696

7 POINTS

P	QUANTILE	BETA APPROX
0.05	0.311873094	0.352245852
0.1	0.35742621	0.3918986[illegible]
0.25	0.434456393	0.4623218
0.5	0.520479724	0.545316264
0.75	0.61203054	0.63085608
0.9	0.69480852	0.706967875
0.95	0.74194579	0.75059561
0.99	0.821151294	0.82511095
0.995	0.84635596	0.849207446
0.999	0.89112524	0.89246614
0.9995	0.9058967	0.9068933

8 POINTS

P	QUANTILE	BETA APPROX
0.05	0.30932574	0.34049277
0.1	0.348457858	0.375207605
0.25	0.414121196	0.43750529
0.5	0.488581225	0.511914775
0.75	0.57107501	0.59055762
0.9	0.64937166	0.66288046
0.95	0.69559435	0.70564512
0.99	0.776561305	0.78163104
0.995	0.803424396	0.80723652
0.999	0.85294489	0.85494204
0.9995	0.86993174	0.871461436

9 POINTS

P	QUANTILE	BETA APPROX
0.05	0.302546[illegible]	0.32694[illegible]
0.1	0.336499736	0.355099922
0.25	0.39357365	0.41404299
0.5	0.45991377	0.481793925
0.75	0.53520826	0.554781474
0.9	0.60926489	0.6235681
0.95	0.654181[illegible]	0.66517882
0.99	0.735333964	0.74128775
0.995	0.76309447	0.7676968[illegible]
0.999	0.81568198	0.818271205
0.9995	0.83423762	0.83627276

10 POINTS

P	QUANTILE	BETA APPROX
0.05	0.29398684	0.313277766
0.1	0.3237205	0.341527507
0.25	0.374072596	0.392512843
0.5	0.43421416	0.454857394
0.75	0.50389310	0.52303557
0.9	0.57379679	0.58851962
0.95	0.617217585	0.62882762
0.99	0.69757704	0.70421176
0.995	0.725718066	0.730966136
0.999	0.76015188	0.783257045
0.9995	0.79977374	0.80227045

5

STRETCH

[illegible] POINTS

P	QUANTILE
0.05	0.3425[illegible]
0.1	0.416[illegible]
0.25	0.5458[illegible]
0.5	0.686[illegible]
0.75	0.80[illegible]
0.9	0.88[illegible]
0.95	0.92355[illegible]
0.99	0.96[illegible]3181
0.995	0.97[illegible]
0.999	0.98989825
0.9995	0.99287[illegible]

6 POINTS

P	QUANTILE	BETA APPROX
0.05	0.36605601	0.43025355
0.1	0.42[illegible]102	0.48509077
0.25	0.53806643	0.57859282
0.5	0.6[illegible]644	0.6803451
0.75	0.75999884	0.77356963
0.9	0.838[illegible]	0.84530977
0.95	0.8[illegible]531	0.88132624
0.99	0.93[illegible]752	0.93366[illegible]975
0.995	0.94[illegible]	0.94795279
0.999	0.96985392	0.97008757
0.9995	0.9[illegible]22828	0.97637228

7 POINTS

P	QUANTILE	BETA APPROX
0.05	0.370[illegible]497	0.43893866
0.1	0.424093768	0.483596[illegible]
0.25	0.51526122	0.560279[illegible]
0.5	0.61498884	0.64611487
0.75	0.71914[illegible]	0.72912649
0.9	0.78748278	0.79798846
0.95	0.82831626	0.8351712[illegible]
0.99	0.891[illegible]941	0.89439063
0.995	0.91046099	0.91224818
0.999	0.941[illegible]1491	0.94244723
0.9995	0.951386966	0.951[illegible]

8 POINTS

P	QUANTILE	BETA APPROX
0.05	0.36730437	0.428603694
0.1	0.41359	0.466940448
0.25	0.490870044	0.5334373
0.5	0.57711749	0.6096416
0.75	0.664308116	0.68618636
0.9	0.7395?018	0.75287013
0.95	0.78132014	0.79056505
0.99	0.85003142	0.854151286
0.995	0.87156538	0.874516055
0.999	0.909467265	0.9108644
0.9995	0.92188124	0.92290262

9 POINTS

P	QUANTILE	BETA APPROX
0.05	0.36202006	0.41279368
0.1	0.402073424	0.446699664
0.25	0.468159243	0.50604968
0.5	0.5430942	0.57532172
0.75	0.623415515	0.6468511
0.9	0.696116015	0.71135287
0.95	0.737919375	0.74898867
0.99	0.80960802	0.81504397
0.995	0.83302359	0.83708052
0.999	0.87577204	0.87788443
0.9995	0.89031939	0.89193015

10 POINTS

P	QUANTILE	BETA APPROX
0.05	0.353007834	0.3957334
0.1	0.388100192	0.42629198[illegible]
0.25	0.446033046	0.4801917[illegible]
0.5	0.512[illegible]24444	0.5440183[illegible]
0.75	0.58713011	0.61131911
0.9	0.65704684	0.6735873[illegible]
0.95	0.698362865	0.710[illegible]4
0.99	0.[illegible]143912	0.[illegible]98419
0.995	0.796047725	0.8010[illegible]4
0.999	0.84219603	0.845002[illegible]
0.9995	0.85834555	0.860544[illegible]

6

STRETCH

6 POINTS

P	QUANTILE
0.05	0.41819[illegible]
0.1	0.489[illegible]83673
0.25	0.6 0520884
0.5	0.73555045
0.75	0.838837184
0.9	0.9[illegible]0447
0.95	0.93715052
0.99	0.9732366
0.995	0.98127894
0.999	0.99174456
0.9995	0.99418119

[illegible] POINTS

P	QUANTILE	BETA APPROX
0.05	0.[illegible]2927508	0.498192355
0.1	0.49114947	0.54989104
0.25	0.59416632	0.63588385
0.5	0.70159964	0.72690152
0.75	0.[illegible]9521231	0.808348216
0.9	0.86376338	0.86994032
0.95	0.8969293	0.90054087
0.99	0.94355349	0.94465117
0.995	0.95595888	0.95652646
0.999	0.974902675	0.97511725
0.9995	0.980223216	0.980355784

8 POINTS

P	QUANTILE	BETA APPROX
0.05	0.42478423	0.50233607
0.1	0.47802405	0.5445085
0.25	0.56629042	0.61556582
0.5	0.66069178	0.6933742
0.75	0.748015[illegible]25	0.767152354
0.9	0.81709246	0.82740[illegible]64
0.95	0.852963015	0.85961203
0.99	0.90794519	0.9104800[illegible]
0.995	0.92401842	0.92572074
0.999	0.95070128	0.95139746
0.9995	0.95892291	0.959400[illegible]

9 POINTS

P	QUANTILE	BETA APPROX
0.05	0.415598437	0.48896737
0.1	0.461694285	0.52541976
0.25	0.537991084	0.58765082
0.5	0.621980235	0.65763048
0.75	0.70391326	0.7266936[illegible]5
0.9	0.77256064	0.78598933
0.95	0.80996947	0.8191857[illegible]
0.99	0.87065463	0.87470011
0.995	0.88947252	0.89235834
0.999	0.92238192	0.92374091
0.9995	0.933104075	0.93409494

10 POINTS

P	QUANTILE	BETA APPROX
0.05	0.40526919	0.47074[illegible]794
0.1	0.44547422	0.5032[illegible]4894
0.25	0.51184325	0.55929712
0.5	0.58701567	0.62364439
0.75	0.66398195	0.68902354
0.9	0.73142581	0.74717192
0.95	0.76949744	0.780[illegible]8513
0.99	0.833810374	0.8392653[illegible]
0.995	0.85458331	0.85863833
0.999	0.89224295	0.894342944
0.9995	0.904986896	0.90658621

7

STRETCH

7 POINTS

P	QUANTILE
0.05	0.479297206
0.1	0.54743533
0.25	0.65928968
0.5	0.7[illegible]1509685
0.75	0.862[illegible]
0.9	0.92117646
0.95	0.9466243[illegible]
0.99	0.[illegible]7733454
0.995	0.984155215
0.999	0.99301857
0.9995	0.995[illegible]80516

8 POINTS

P	QUANTILE	BETA APPROX
0.05	0.4[illegible]18521	0.552020594
0.1	0.54136138	0.600398585
0.25	0.63836[illegible]22	0.67948107
0.5	0.73715834	0.76156287
0.75	0.82137732	0.83379893
0.9	0.88[illegible]9191	0.88775115
0.95	0.91101126	0.9143577
0.99	0.95148424	0.952493235
0.995	0.96219115	0.96280341
0.999	0.978492305	0.978686754
0.9995	0.983060405	0.983182475

9 POINTS

P	QUANTILE	BETA APPROX
0.05	0.471458957	0.55302577
0.1	0.523273036	0.592619464
0.25	0.60774692	0.6584373
0.5	0.696761645	0.72937827
0.75	0.77706294	0.79569296
0.9	0.839369334	0.84923701
0.95	0.871338405	0.87765[illegible]6
0.99	0.91[illegible]76	0.92225785
0.995	0.933956237	0.93556074
0.999	0.957255885	0.95[illegible]
0.9995	0.96441034	0.96485665

10 POINTS

P	QUANTILE	BETA APPROX
0.05	0.45772414	0.53784804
0.1	0.50309138	0.57229[illegible]62
0.25	0.57761435	0.63044[illegible]23
0.5	0.658304736	0.69492269
0.75	0.73504214	0.[illegible]77557[illegible]
0.9	0.797786545	0.811120555
0.95	0.8318439	0.84078364
0.99	0.886189975	0.89007239
0.995	0.90290885	0.905672595
0.999	0.93200831	0.93330531
0.9995	0.9414535	0.94239905

8

STRETCH

8 POINTS

P	QUANTILE
0.05	0.529320285
0.1	0.593754336
0.25	0.69730048
0.5	0.79886869
0.75	0.87937121
0.9	0.93137412
0.95	0.953610934
0.99	0.98034243
0.995	0.9862638
0.999	0.99395136
0.9995	0.9957376

9 POINTS

P	QUANTILE	BETA APPROX
0.05	0.52600627	0.59560351
0.1	0.58284621	0.64080958
0.25	0.6740108	0.71376376
0.5	0.76516585	0.788388774
0.75	0.84159522	0.85325007
0.9	0.895895526	0.90124659
0.95	0.9216886	0.92478423
0.99	0.95744947	0.95837836
0.995	0.96686892	0.96743158
0.999	0.981178805	0.98135905
0.9995	0.98518138	0.985292956

10 POINTS

P	QUANTILE	BETA APPROX
0.05	0.51155715	0.59442762
0.1	0.561626956	0.631562[illegible]
0.25	0.642479464	0.69267134
0.5	0.72592883	0.75775485
0.75	0.80006842	0.81793932
0.9	0.856762454	0.86611704[illegible]
0.95	0.88558917	0.891544856
0.99	0.929036655	0.93127207
0.995	0.94158129	0.9430757
0.999	0.96225791	0.962864436
0.9995	0.96859221	0.96900897

9

STRETCH

9 POINTS

P	QUANTILE
0.05	0.570864245
0.1	0.631638095
0.25	0.72773032
0.5	0.820380725
0.75	0.89283709
0.9	0.93923049
0.95	0.95897727
0.99	0.98264365
0.995	0.98787646
0.999	0.99466376
0.9995	0.996240176

10 POINTS

P	QUANTILE	BETA APPROX
0.05	0.56348471	0.63155703
0.1	0.61761622	0.67384963
0.25	0.70332769	0.741425075
0.5	0.78778987	0.80977396
0.75	0.857691325	0.86860804
0.9	0.90685706	0.91183142
0.95	0.93006854	0.93293719
0.99	0.96210245	0.96295981
0.995	0.970512904	0.97103171
0.999	0.983265445	0.98343234
0.9995	0.986828364	0.98693137

10

STRETCH

10 POINTS

P	QUANTILE
0.05	0.605836436
0.1	0.663152255
0.25	0.752629794
0.5	0.837737605
0.75	0.903596446
0.9	0.94547133
0.95	0.96322874
0.99	0.984462306
0.995	0.98914961
0.999	0.99522547
0.9995	0.996636905

Table 3. Formulas for Computing the A^{th} Moment Around the Origin of the Maximum p-Stretch. *

$$\boxed{\text{STRETCH}^{2}}$$

2 PTS. : 2

3 PTS. : $12 - \frac{6}{A_2}$

4 PTS. : $\frac{24}{A_3} - \frac{72}{A_2} + 72$

5 PTS. : $- \frac{120}{A_4} + \frac{480}{A_3} - \frac{720}{A_2} + 480$

6 PTS. : $\frac{720}{A_5} - \frac{3600}{A_4} + \frac{7200}{A_3} - \frac{7200}{A_2} + 3600$

7 PTS. : $- \frac{5040}{A_6} + \frac{30240}{A_5} - \frac{75600}{A_4} + \frac{100800}{A_3} - \frac{75600}{A_2} + 30240$

8 PTS. : $\frac{40320}{A_7} - \frac{282240}{A_6} + \frac{846720}{A_5} - \frac{1411200}{A_4} + \frac{1411200}{A_3} - \frac{846720}{A_2} + 282240$

9 PTS. : $- \frac{362880}{A_8} + \frac{2903040}{A_7} - \frac{10160640}{A_6} + \frac{20321280}{A_5} - \frac{25401600}{A_4} + \frac{20321280}{A_3} - \frac{10160640}{A_2} + 2903040$

10 PTS. : $\frac{3628800}{A_9} - \frac{32659200}{A_8} + \frac{130636800}{A_7} - \frac{304819200}{A_6} + \frac{457228800}{A_5} - \frac{457228800}{A_4} + \frac{304819200}{A_3} - \frac{130636800}{A_2} + 32659200$

*Each listed formula must be multiplied by $\prod_{i=0}^{k} \frac{1}{A+i}$ to achieve the A^{th} moment around the origin for k points (see page 18).

3 STRETCH

3 PTS. : $6 A + 6$

4 PTS. : $\frac{24}{A_2} + 48 A$

5 PTS. : $- \frac{15 A^2}{A_2} - \frac{15 A}{A_2} + \frac{240}{A_2} + 360 A - 120$

6 PTS. : $\frac{1440}{A_3} - \frac{360 A^2}{A_2} + \frac{360 A}{A_2} + \frac{720}{A_2} + 2880 A - 1440$

7 PTS. : $\frac{280 A^3}{A_{9\,3}} - \frac{560 A^2}{A_{3\,3}} - \frac{17080 A}{A_{9\,3}} + \frac{25200}{A_3} - \frac{5670 A^2}{A_2} + \frac{9450 A}{A_2} - \frac{5040}{A_2}$

$+ 25200 A - 15120$

8 PTS. : $\frac{201600}{A_4} + \frac{17920 A^3}{A_{9\,3}} - \frac{35840 A^2}{A_{3\,3}} + \frac{116480 A}{A_{9\,3}} + \frac{161280}{A_3} - \frac{80640 A^2}{A_2}$

$+ \frac{161280 A}{A_2} - \frac{161280}{A_2} + 241920 A - 161280$

9 PTS. :

$$- \frac{945 A^4}{16 A_4} + \frac{12285 A^3}{8 A_4} - \frac{101115 A^2}{16 A_4} - \frac{2966355 A}{8 A_4} + \frac{5080320}{A_4} + \frac{60480 A^3}{A_3}$$

$$- \frac{362880 A^2}{A_3} + \frac{1028160 A}{A_3} - \frac{1134000 A^2}{A_2} + \frac{2494800 A}{A_2} - \frac{2903040}{A_2} + 2540160 A$$

$$- 1814400$$

10 PTS. :

$$\frac{50803200}{A_5} - \frac{9450 A^4}{A_4} + \frac{170100 A^3}{A_4} - \frac{1237950 A^2}{A_4} - \frac{1417500 A}{A_4} + \frac{50803200}{A_4}$$

$$+ \frac{1433600 A^3}{A_3} - \frac{8601600 A^2}{A_3} + \frac{28672000 A}{A_3} - \frac{29030400}{A_3} - \frac{16329600 A^2}{A_2} + \frac{38102400 A}{A_2}$$

$$- \frac{47174400}{A_2} + 29030400 A - 21772800$$

$$\boxed{\text{STRETCH}^4}$$

4 PTS. : $12A^2 + 36A + 24$

5 PTS. : $-\frac{120}{2^A} + 120A^2 + 120A + 240$

6 PTS. : $\frac{90A^2}{2^A} + \frac{450A}{2^A} - \frac{720}{2^A} + 1080A^2 + 360A + 1440$

7 PTS. : $-\frac{105A^4}{4\,2^A} - \frac{315A^3}{2\,2^A} + \frac{3885A^2}{4\,2^A} + \frac{12285A}{2\,2^A} - \frac{5040}{2^A} + 10080A^2 + 10080$

8 PTS. : $\frac{201600}{3^A} - \frac{945A^4}{2^A} - \frac{1470A^3}{2^A} - \frac{7875A^2}{2^A} + \frac{153930A}{2^A} - \frac{241920}{2^A} + 100800A^2$

$- 20160A + 80640$

9 PTS. : $\frac{4480A^3}{3^A} + \frac{13440A^2}{3^A} - \frac{716800A}{3^A} - \frac{20790A^4}{2^A} - \frac{3780A^3}{2^A} - \frac{228690A^2}{2^A}$

$+ \frac{1568700A}{2^A} - \frac{362880}{2^A} + 1088640A^2 - 362880A + 725760$

10 PTS. : $\frac{2800A^6}{81\,3^A} - \frac{2800A^5}{27\,3^A} - \frac{417200A^4}{81\,3^A} + \frac{176400A^3}{3^A} + \frac{159880000A^2}{81\,3^A}$

$- \frac{180219200A}{27\,3^A} - \frac{387450A^4}{2^A} + \frac{170100A^3}{2^A} - \frac{4035150A^2}{2^A} + \frac{17180100A}{2^A} - \frac{3628800}{2^A}$

$+ 12700800A^2 - 5443200A + 7257600$

5

STRETCH

$$5 \text{ PTS. : } 20 A^{3} + 120 A^{2} + 220 A + 120$$

$$6 \text{ PTS. : } \frac{720}{A^{2}} + 240 A^{3} + 720 A^{2} + 1920 A$$

$$7 \text{ PTS. : } - \frac{630 A^{2}}{A^{2}} - \frac{5670 A}{A^{2}} - \frac{5040}{A^{2}} + 2520 A^{3} + 5040 A^{2} + 12600 A + 10080$$

$$8 \text{ PTS. : } \frac{210 A^{4}}{A^{2}} + \frac{2940 A^{3}}{A^{2}} + \frac{7350 A^{2}}{A^{2}} - \frac{35700 A}{A^{2}} - \frac{40320}{A^{2}} + 26880 A^{3} + 40320 A^{2}$$

$$+ 94080 A + 80640$$

$$9 \text{ PTS. : } - \frac{315 A^{6}}{8 A^{2}} - \frac{4725 A^{5}}{8 A^{2}} + \frac{3465 A^{4}}{8 A^{2}} + \frac{352485 A^{3}}{8 A^{2}} + \frac{667485 A^{2}}{4 A^{2}} - \frac{239085 A}{A^{2}}$$

$$- \frac{362880}{A^{2}} + 302400 A^{3} + 362880 A^{2} + 786240 A + 725760$$

$$10 \text{ PTS. : } \frac{50803200}{A^{3}} - \frac{1890 A^{6}}{A^{2}} - \frac{15120 A^{5}}{A^{2}} - \frac{103950 A^{4}}{A^{2}} + \frac{1096200 A^{3}}{A^{2}} - \frac{1595160 A^{2}}{A^{2}}$$

$$+ \frac{18990720 A}{A^{2}} - \frac{54432000}{A^{2}} + 3628800 A^{3} + 3628800 A^{2} + 7257600 A + 7257600$$

6

STRETCH

6 PTS. : $30A^4 + 300A^3 + 1050A^2 + 1500A + 720$

7 PTS. : $-\frac{5040}{A_2} + 420A^4 + 2520A^3 + 9660A^2 + 7560A + 10080$

8 PTS. : $\frac{5040A^2}{A_2} + \frac{65520A}{A_2} + \frac{161280}{A_2} + 5040A^4 + 23520A^3 + 75600A^2 + 137760A - 120960$

9 PTS. : $-\frac{1890A^4}{A_2} - \frac{41580A^3}{A_2} - \frac{292950A^2}{A_2} - \frac{616140A}{A_2} - \frac{362880}{A_2} + 60480A^4 + 241920A^3 + 665280A^2 + 1209600A + 725760$

10 PTS. : $\frac{1575A^6}{A_2^4} + \frac{42525A^5}{A_2^4} + \frac{360675A^4}{A_2^4} + \frac{335475A^3}{A_2^4} - \frac{4093425A^2}{A_2^2} - \frac{5679450A}{A_2} - \frac{3628800}{A_2} + 756000A^4 + 2721600A^3 + 6501600A^2 + 11793600A + 7257600$

$$\boxed{STRETCH^{7}}$$

7 PTS. : $42 A^{5} + 630 A^{4} + 3570 A^{3} + 9450 A^{2} + 11508 A + 5040$

8 PTS. : $\dfrac{40320}{A_{2}} + 672 A^{5} + 6720 A^{4} + 36960 A^{3} + 73920 A^{2} + 123648 A$

9 PTS. : $- \dfrac{45360 A^{2}}{A_{2}} - \dfrac{771120 A}{A_{2}} - \dfrac{2903040}{A_{2}} + 9072 A^{5} + 75600 A^{4} + 347760 A^{3}$

$+ 1013040 A^{2} - 356832 A + 3265920$

10 PTS. : $\dfrac{18900 A^{4}}{A_{2}} + \dfrac{567000 A^{3}}{A_{2}} + \dfrac{6104700 A^{2}}{A_{2}} + \dfrac{27329400 A}{A_{2}} + \dfrac{47174400}{A_{2}}$

$+ 120960 A^{5} + 907200 A^{4} + 3628800 A^{3} + 9979200 A^{2} + 14394240 A - 43545600$

$$\boxed{\text{STRETCH}^{8}}$$

$$8\ \text{PTS.}:\ 56 A^{6} + 1176 A^{5} + 9800 A^{4} + 41160 A^{3} + 90944 A^{2} + 98784 A + 40320$$

$$9\ \text{PTS.}:\ - \frac{362880}{A_{2}} + 1008 A^{6} + 15120 A^{5} + 115920 A^{4} + 408240 A^{3} + 971712 A^{2} + 665280 A + 725760$$

$$10\ \text{PTS.}:\ \frac{453600 A^{2}}{A_{2}} + \frac{9525600 A}{A_{2}} + \frac{47174400}{A_{2}} + 15120 A^{6} + 196560 A^{5} + 1285200 A^{4} + 5367600 A^{3} + 4142880 A^{2} + 32538240 A - 43545600$$

$$\boxed{\text{STRETCH}^{9}}$$

$$9\ \text{PTS.}:\ 72 A^{7} + 2016 A^{6} + 23184 A^{5} + 141120 A^{4} + 487368 A^{3} + 945504 A^{2} + 940896 A + 362880$$

$$10\ \text{PTS.}:\ \frac{3628800}{A_{2}} + 1440 A^{7} + 30240 A^{6} + 312480 A^{5} + 1663200 A^{4} + 5664960 A^{3} + 9192960 A^{2} + 12165120 A$$

$$\boxed{\text{STRETCH}^{10}}$$

$$10\ \text{PTS.}:\ 90 A^{8} + 3240 A^{7} + 49140 A^{6} + 408240 A^{5} + 2020410 A^{4} + 6055560 A^{3} + 10631160 A^{2} + 9862560 A + 3628800$$

Table 4. Numerican Values of the Moments Around the Origin.

2 STRETCH

```
2  PTS. : [0.333333333, 0.166666666, 0.1, 0.066666666, 0.04761[illegible]6]
3  PTS. : [0.375, 0.174999999, 0.09375, 0.0553571433, 0.03515625[illegible]5]
4  PTS. : [0.366666667, 0.157407407, 0.076058201, 0.040354938, 0.0230981363]
5  PTS. : [0.34[illegible]22222, 0.137235448, 0.060402198, 0.02[illegible]765, 0.0151904[illegible]]
6  PTS. : [0.32619047, 0.119236111, 0.0482172053, 0.021335835, 0.0102253014]
7  PTS. : [0.30625, 0.104081767, 0.0389561323, 0.0159383[illegible], [illegible]07158[illegible]E-3]
8  PTS. : [0.288095236, 0.09149672[illegible], 0.0318965409, 0.0121444[illegible], 5.01902276E-3]
9  PTS. : [0.27178571[illegible]4, 0.081037902, 0.0264548396, 9.4251[illegible]91E-3, 3.69728153E-3]
10 PTS. : [0.25717894[illegible], 0.072294159, 0.0222035733, 7.4371461E-3, 2.70697084E-3]
```

3 STRETCH

```
3  PTS. : [0.5, [illegible], [illegible], 0.142857144, 0.107142858]
4  PTS. : [0.5, 0.283333333, 0.175, 0.115178572, 0.0796130095]
5  PTS. : [0.479166666, 0.25297619, 0.143973215, 0.087053571, 0.0553385415]
6  PTS. : [0.45238095, 0.222222222, 0.116953261, 0.065296148, 0.0383700863]
7  PTS. : [0.42573302, 0.19509459, 0.095813949, 0.049460438, 0.02[illegible]1389493]
8  PTS. : [0.40089162[illegible], 0.[illegible]72010742, 0.078560941, 0.0380106173, 0.0193942245]
9  PTS. : [0.378287[illegible]62, 0.152558299, 0.065366504, 0.0296592333, 0.01420211[illegible]]
10 PTS. : [0.35788[illegible]9, 0.136156479, 0.0549362997, 0.0234831378, 0.0105960063]
```

4 STRETCH

```
4  PTS. : [0.6, 0.4, 0.285714287, 0.214285716, 0.166666666]
5  PTS. : [0.583333336, 0.369047762, 0.247767856, 0.174107132, 0.12686012[illegible]]
6  PTS. : [0.55357143, 0.328125, 0.205357142, 0.134375, 0.0912642051]
7  PTS. : [0.5234375, 0.29079861, 0.169921873, 0.103716855, 0.065[illegible]650[illegible]2]
8  PTS. : [0.49421296, 0.257831786, 0.141131802, 0.080658592, 0.0479229395]
9  PTS. : [0.46705246, 0.229449583, 0.118077666, 0.063425469, 0.0354454056]
10 PTS. : [0.44228695, 0.20523516, 0.099642827, 0.0504884436, 0.0266324371]
```

5

STRETCH

5 PTS. : [0.666666664, 0.476190478, 0.35714286, 0.27777776, 0.22222222]
6 PTS. : [0.64285714, 0.4375, 0.311011907, 0.228869047, 0.17322781[illegible]]
7 PTS. : [0.609375, 0.390625004, 0.260937497, 0.180397727, 0.128403172]
8 PTS. : [0.57638889, 0.347916663, 0.218465907, 0.141966539, 0.0950[illegible]]
9 PTS. : [0.54609375, 0.3111505[illegible], 0.184126416, 0.112734921, 0.071[illegible]41861]
10 PTS. : [0.51799242, 0.27925084, 0.156178167, 0.090[illegible]3723, 0.0539535[illegible]]

6

STRETCH

6 PTS. : [0.7[illegible], 0.53571429, 0.41666664, 0.333[illegible]2, 0.272727273]
7 PTS. : [0.6875, 0.48055552, 0.365624998, 0.27859848, 0.217[illegible]]
8 PTS. : [0.652[illegible], 0.44305555, 0.31060606, 0.2238005, 0.165[illegible]52]
9 PTS. : [0.61875, 0.39715909, 0.263137207, 0.179250436, 0.125136[illegible]]
10 PTS. : [0.58735 95, 0.357185136, 0.22406305, 0.144542957, 0.09563[illegible]1093]

7

STRETCH

7 PTS. : [0.75000[illegible], 0.5[illegible]33336, 0.46666666, 0.36[illegible], 0.3[illegible]]
8 PTS. : [0.72222222, 0.53998[illegible], 0.[illegible]26879, 0.33[illegible]5858, 0.25798[illegible]]
9 PTS. : [0.69[illegible]9993, 0.48749[illegible]3, 0.35482954, 0.2[illegible], 0.20045[illegible]]
10 PTS. : [0.6[illegible]40909, 0.43986[illegible]426, 0.303922642, 0.214[illegible]1064, 0.1150480[illegible]]

8

STRETCH

8 PTS. : [0.7[illegible]776, 0.62222[illegible], 0.50909091, 0.42424242, 0.358974356]
9 PTS. : [0.749999985, 0.57727[illegible], 0.453977268, 0.36[illegible], 0.295[illegible]]
10 PTS. : [0.[illegible]590909?, 0.52556818, 0.394230768, 0.30129246, 0.234070614]

9

STRETCH

```
[illegible] PTS. :  10.8, 0.6[illegible]456, 0.[illegible]45455. 0.461538[illegible], 0.3[illegible]043943
10  PTS. : [illegible], 0.[illegible]9848484, 0.4[illegible]7554, 0.399[illegible], 0.[illegible]
```

10

STRETCH

```
[illegible]  PTS. :  10.8181[illegible], 0.[illegible]81917. 0.5[illegible]2308, 0.4945055. 0.42[illegible]
[illegible]                                                DONE

[illegible]
In editor:
[illegible] NS:[illegible] THRU 10 D[illegible]
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295

EXACT DISTRIBUTIONS FOR GAPS AND STRETCHES

Exact distributions for the largest gaps and stretches (higher order spacings) from points uniformly distributed on the unit interval are given, for up to ten points. General moment formulas and quantiles from these distributions, as well as the distributions themselves, are tabled, with approximate quantiles (obtained from an independent-beta approximation) for comparison.

The results are derived via a recursive formulation, which is then implemented on the MACSYMA symbolic omputation system. Moment generating functions derived by the same approach are also given.

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